MATH 2050C Lecture on 3/13/2020

[Announcement: PS 6 due today, PS 7 posted.]

Common Mistake in PS 5:

$$\underbrace{\lim\left(\frac{(-1)^n n}{n+1}\right)}_{\text{Convergent?}} = \lim\left(\frac{(-1)^n}{1+\frac{1}{n}}\right) \overset{\text{wrong}}{=} \frac{\lim((-1)^n)}{\lim(1+\frac{1}{n})} = \lim((-1)^n) \text{ does not exist}$$

$$\Rightarrow \lim\left(\frac{(-1)^n n}{n+1}\right) \text{ does not exist.}$$

$\because$ we do not know the limits exist.

Recall: (x_n) convergent $\Rightarrow$ (x_n) bdd

$\Leftarrow$ & (x_n) monotone

Q: What can we say without monotonicity?

* Bolzano-Weierstrass Thm *: (x_n) bdd $\Rightarrow$ $\exists$ subseq. (x_{n_k}) convergent.

Remark: There could be two subseq's (x_{n_k}), $(x_{n_{k'}})$ converging to different limits ($\Rightarrow$ (x_n) divergent).

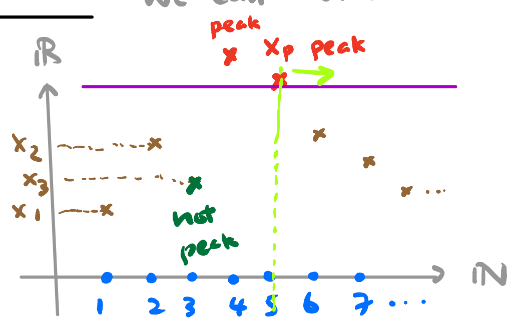
Proof: (Constructive proof)

Claim: $\exists$ monotone subseq. (x_{n_k}) of (x_n) (by MCT, (x_{n_k}) convergent)

i.e. either $x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \cdots$

or $x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq \cdots$

Pf of Claim: We can "visualize" the graph of a seq. $X = (x_n) : \mathbb{N} \to \mathbb{R}$

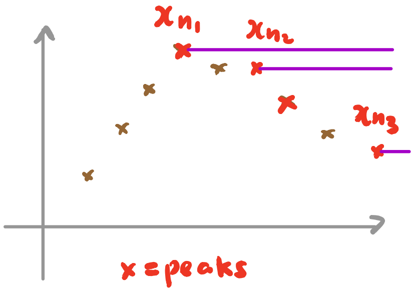


Def^{n}: We say that the p^{th}-term x_p of (x_n) is a peak if

$$x_k \leq x_p \quad \forall\, k \geq p$$

Given a seq. (x_n), there are 2 possible scenarios:

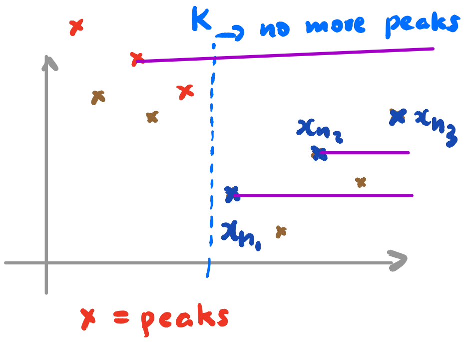
Scenario 1 : $\exists$ infinitely many peaks.



$x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq \cdots$ decreasing subseq.

$\uparrow$ $\because x_{n_1}$ peak $\uparrow$ $\because x_{n_2}$ peak

<u>Scenario 2</u> : $\exists$ finitely many (could be none) peaks.



$\exists K \in \mathbb{N}$ s.t. x_n is <u>NOT</u> a peak $\forall n \geq K$

Take $x_{n_1} = x_K$, $\because x_K$ is <u>NOT</u> a peak

$\exists n_2 > n_1$ s.t. $x_{n_2} > x_{n_1}$

Similarly, $\because x_{n_2}$ is <u>NOT</u> a peak,

$\exists n_3 > n_2$ s.t. $x_{n_3} > x_{n_2}$

So, $x_{n_1} < x_{n_2} < x_{n_3} < \cdots\cdots$ increasing subseq.

$\square$

<u>Remark:</u> BWT relates to the "compactness" of a closed bdd interval in $\mathbb{R}$.
Also, try to prove using "nested interval property". (Ex.)

<u>Thm:</u> Let (x_n) be a bdd seq.

(x_n) converges to x $\iff$ $\underbrace{\forall \text{ convergent subseq. } (x_{n_k}) \text{ converges to the same } x}_{(*)}$

<u>Proof:</u> "$\Rightarrow$" DONE.

"$\Leftarrow$" Prove by contradiction. Suppose (x_n) <u>does NOT</u> converge to x.

$\Rightarrow \exists\, \varepsilon_0 > 0$ and a subseq. $(x_{n_k})_k$ of $(x_n)_n$ s.t.

(#) $\cdots\cdots$ $|x_{n_k} - x| \geq \varepsilon_0 \quad \forall k \in \mathbb{N}$.

Now, since $(x_{n_k})_k$ is bdd ($\because (x_n)_n$ bdd)

by BWT, $\exists$ a further subseq. $(x_{n_{k_\ell}})_\ell$ convergent.

Using $(*)$ ($\because (x_{n_{k_\ell}})_\ell$ is still a subseq. of $(x_n)_n$)

$$\lim_{\ell \to \infty} (x_{n_{k_\ell}}) = x$$

contradictory!

BuT (#) $\Rightarrow$ $|x_{n_{k_\ell}} - x| \geq \varepsilon_0 \quad \forall \ell \in \mathbb{N}$

$\square$

Cauchy Criteria (§ 3.5 in textbook)

Q: When is (x_n) convergent without "knowing" its limit?

Sufficient condition: monotone + bdd $\Rightarrow$ convergent

Note: $\Leftarrow$ false, e.g. $(x_n) = \left(\frac{(-1)^n}{n}\right) \to 0$

Necessary & Sufficient Condition: "Cauchy" $\Leftrightarrow$ convergent.

Def$^{\underline{n}}$: A seq. (x_n) is Cauchy

if $\forall \varepsilon > 0$, $\exists H = H(\varepsilon) \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \varepsilon \qquad \forall n, m \geq H$$

Remark: There is no possible "limit" needed in the definition!

Example 1: $(x_n) = \left(\frac{1}{n}\right)$ is Cauchy.

Pf: Let $\varepsilon > 0$. Choose $H \in \mathbb{N}$ s.t. $H > \frac{2}{\varepsilon}$.

Then, $\forall n, m \geq H$, we have

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{H} + \frac{1}{H} = \frac{2}{H} < \varepsilon.$$

$\square$

Example 2: $(x_n) = (1 + (-1)^n)$ is NOT Cauchy.

Pf: (x_n) is NOT Cauchy $\Leftrightarrow$ $\exists \varepsilon_0 > 0$ s.t. $\forall H \in \mathbb{N}$, $\exists m, n \geq H$ with

$$|x_m - x_n| \geq \varepsilon_0$$

Note: $(x_n) = (0, 2, 0, 2, 0, 2, 0, 2, \ldots\ldots)$

Take $\varepsilon_0 = 1 > 0$. For any $H \in \mathbb{N}$ fixed, $\exists m, n \geq H$ s.t. m is odd, n is even.

but $|x_m - x_n| = |0 - 2| = 2 \geq 1 = \varepsilon_0$

$\square$

* **Thm:** (x_n) convergent $\Longleftrightarrow$ (x_n) Cauchy *

Proof: "$\Rightarrow$" ✓ (Easier)

Assume (x_n) is convergent, say $\lim(x_n) = x$.

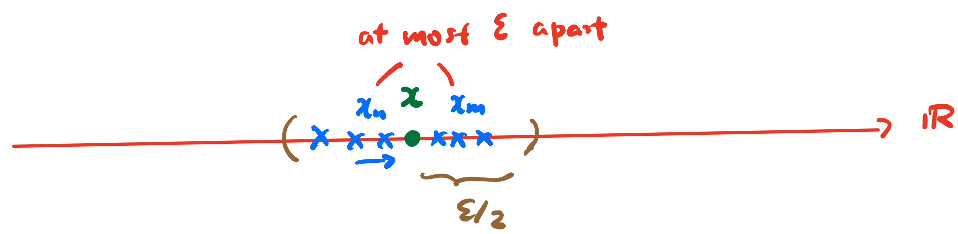
By def$^{\underline{n}}$ of limit, $\underbrace{\forall \varepsilon > 0,\ \exists K \in \mathbb{N} \text{ s.t. } |x_n - x| < \varepsilon \quad \forall n \geq K}_{(\#)}$

Claim: (x_n) is Cauchy.

Pf: Let $\varepsilon > 0$. By $(\#)$, $\exists \overset{K(\varepsilon/2)}{K''} \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon/2 \quad \forall n \geq K$

Take $H := K \in \mathbb{N}$. Then, $\forall\ m, n \geq H$,

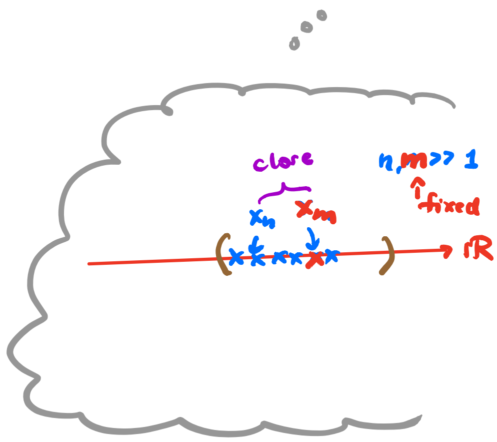
$$|x_m - x_n| \leq |x_m - x| + |x_n - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$ $\square$



"$\Leftarrow$" (More difficult)

Assume (x_n) is Cauchy.

✓ Step 1: (x_n) is bdd.



Take $\varepsilon_0 = 1 > 0$, by def$^{\underline{n}}$ of Cauchy seq.,

$\exists\ H = H(1) \in \mathbb{N}$ s.t.

$$|x_n - x_m| < 1 = \varepsilon_0 \quad \forall n, m \geq H$$

In particular, fix $m = H$, then

$$|x_n - x_H| < 1 \quad \forall n \geq H$$

$$\Rightarrow \quad |x_n| < 1 + |x_H| \quad \forall n \geq H$$

Then, $|x_n| \leq M := \max\{|x_1|, \dots, |x_{H-1}|, 1 + |x_H|\} \quad \forall n \in \mathbb{N}$.

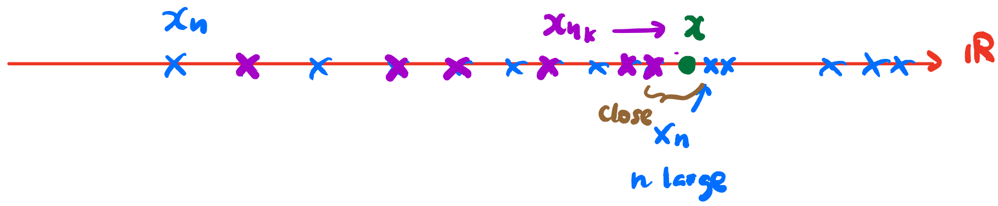
So, (x_n) is bdd.

Step 2: (x_n) is convergent.

Since (x_n) is bdd by Step 1, by BWT,

$\exists$ subseq. $(x_{n_k}) \longrightarrow x$ for some $x \in \mathbb{R}$

Want to Show: $\lim (x_n) = x$



Let $\varepsilon > 0$. By Cauchy, $\exists\ H = H(\varepsilon/2) > 0$ s.t.

(1) ———— $|x_m - x_n| < \varepsilon/2 \quad \forall n, m \geq H$

Since $(x_{n_k}) \to x$, by def^{n} of limit, $\exists\ K \in \mathbb{N}$ s.t.

(2) ———— $|x_{n_k} - x| < \varepsilon/2 \quad \forall k \geq K$

Fix $k \geq K$ and $n_k \geq H$. (Ex: why?)

Then $\forall n \geq H$,

$$|x_n - x| \leq \underbrace{|x_n - x_{n_k}|}_{\substack{< \varepsilon/2 \\ \text{by (1), taking} \\ m = n_k}} + \underbrace{|x_{n_k} - x|}_{\substack{< \varepsilon/2 \\ \text{by (2)}}} < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

———— $\square$